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LAGRANGE TRANSFORMATIONS AND DUALITY FOR CORNER AND FLAG SINGULARITIES

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ABSTRACT

Singularities on a space with a fixed collection of subspaces are studied. Homological objects for the singularities are constructed. A Lagrange transformation of the singularities is defined. It is shown that on the set of the isolated singularities, the Lagrange transformation is an involution realizing the duality of corresponding homological objects.

Introduction

Two holomorphic germs $f, g\colon (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ are called **equivalent** if $f \circ h = g$ for some biholomorphic germ $h\colon (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^n, 0)$. A holomorphic germ $\tilde{f}\colon (\mathbb{C}^n \times \mathbb{C}^k, 0) \longrightarrow (\mathbb{C}, 0)$ is called a **stabilization** of f if $\tilde{f} = f + Q$, where $Q\colon (\mathbb{C}^k, 0) \longrightarrow (\mathbb{C}, 0)$ is a germ of a Morse function. Two holomorphic germs $f_i\colon (\mathbb{C}^{n_i}, 0) \longrightarrow (\mathbb{C}, 0)$ are called **stable equivalent** if some their stabilizations are equivalent.

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Definition: A **singularity** is a class of stable equivalent germs at a critical point.

In this paper, we consider only **isolated** singularities, i.e. stable equivalence classes of functional germs at **isolated** critical points.

Additional structures, such as a collection of subspaces of fixed **codimensions**, may be incorporated into the definition of a singularity. This is done by allowing, in the definition of equivalence of germs, only those biholomorphisms which preserve the given structure. Moreover, in order to obtain the stabilization as a germ on a space with the same structure, we must lift the structure from $\mathbb{C}^n$ to $\mathbb{C}^n \times \mathbb{C}^k$ by multiplication by $\mathbb{C}^k$.

In this setting, the set of the corresponding isolated singularities has some symmetry.

The simplest case is the case of boundary singularities (see [1]), i.e. singularities given by germs on a space with some fixed smooth hypersurface called a boundary.

In [6], to each boundary singularity a decomposition into two (ordinary) singularities was associated, and a **Lagrange transformation** on the set of the boundary singularities was described. In the decomposition of a boundary singularity, the first singularity is obtained by neglecting the boundary, the second one is the restriction to the boundary. The Lagrange transformation applied to a boundary singularity gives a new one having the decomposition which is obtained from the original decomposition by the transposition.

In [11], homological objects associated to a boundary singularity were studied. These objects proved to be dual extensions of the corresponding objects associated to the singularities of the decomposition.

In [9], the description of the Lagrange transformation in terms of the duality between the homological objects associated to a boundary singularity was constructed. Thus the set of the isolated boundary singularities has the symmetry which reflects the duality of the corresponding homological objects.

In the present paper, we transfer this result to more complicated structures. One way to generalize "a space with a boundary" is to consider a space with a fixed number of transversal hypersurfaces through the origin, i.e. a space with a corner. The **corner singularities** were studied in [10], [5], [7], [12]. Corner singularities define some objects in symplectic and contact geometry which generalize Lagrange and Legendre submanifolds and arise in different problems of wave propagation theory, calculus of variations, mathematical physics etc. (see, for example, [4]).

Another generalization of a boundary singularity is a singularity on a space with a flag of subspaces of fixed codimensions. Such **flag singularities** appear in the study of projections of complete intersections, in the theory of flattening of curves; they are closely related to symmetric singularities (see [13], [8]).

The structure of the paper is the following. In section 1, we recall definitions of arrays and homological diagrams (sect. 1.1), and discuss their duality (sect. 1.2). In section 2, the homological diagrams for singularities are built. We begin with an isolated singularity (sect. 2.1), then we consider the case of boundary singularities, which are 0-corner singularities in our terminology (sect. 2.2), in sect. 2.3 we generalize the construction of sect. 2.2 for a k-corner singularity. The homological diagram for a k-corner singularity is described as an extension of the homological diagrams of the two $(k-1)$-corner singularities which are given by the original one.

Section 3 is devoted to the duality of the k-corner singularities. In sect. 3.1, we define Lagrange reflections corresponding to the hyperplanes of the k-corner, and describe their action on the decomposition of a k-corner singularity. Then in sect. 3.2 we study the Lagrange transformation which is the product of the $(k+1)$ Lagrange reflections (corresponding to all hyperplanes of a k-corner) and its action on k-corner singularities. We prove that the homological diagrams of a k-corner singularity and its Lagrange transform are dual, up to a shift.

In the last section, we adapt these methods to flag singularities. We describe the Lagrange transformation for this case (sect. 4.1), then we show that the Lagrange transformation of flag singularities gives a geometric realization of the duality for the homological diagrams associated to the flag singularities (sect. 4.2).

1. Arrays and homological diagrams

1.1 ARRAYS. In this section we recall the definition of arrays, and of homological diagrams (see [12]).

Let $\mathcal{C}_\bullet$ (*resp.* $\tilde{\mathcal{C}}^\bullet$) be a complex of free $\mathbb{Z}$-modules with differential d of degree -1 (*resp.* $\tilde{d}$ of degree $+1$).

The dual complex of $\mathcal{C}_\bullet$ (*resp.* $\tilde{\mathcal{C}}^\bullet$), denoted by $D\mathcal{C}^\bullet$ (*resp.* $D\tilde{\mathcal{C}}_\bullet$), is defined by:

$$(D\mathcal{C})^p = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}_p, \mathbb{Z}), \quad (D\tilde{\mathcal{C}})_p = \mathrm{Hom}_{\mathbb{Z}}(\tilde{\mathcal{C}}^p, \mathbb{Z}),$$

with the differential ∂, of degree $+1$ (*resp.* $\tilde{\partial}$ of degree -1), defined by

$$\begin{aligned} \forall \varphi \in D\mathcal{C}^p, \forall a \in \mathcal{C}_{p+1}, \quad \langle \partial\varphi, a\rangle &= (-1)^{p+1}\langle \varphi, da\rangle; \\ \forall \tilde{\varphi} \in D\tilde{\mathcal{C}}_p, \forall \tilde{a} \in \mathcal{C}^{p-1}, \quad \langle \tilde{\partial}\tilde{\varphi}, \tilde{a}\rangle &= (-1)^{p+1}\langle \tilde{\varphi}, \tilde{d}\tilde{a}\rangle. \end{aligned}$$

Definition 1.1: A T-**array** is a diagram $\mathcal{T}$ of the following type:

$$\begin{array}{ccc} C_\bullet & \xleftarrow{v} & D\tilde{C}_\bullet \\ \tilde{C}^\bullet & \xleftarrow{v^*} & DC^\bullet \end{array}$$

where C is a complex of free $\mathbb{Z}$-modules with differential of degree -1, and $\tilde{C}$ is a complex of free $\mathbb{Z}$-modules with differential of degree $+1$, and v and v^* are quasi-isomorphisms.

The complexes C, $\tilde{C}$, DC, $D\tilde{C}$ are called the **vertices** of the T-array. The morphisms v and v^* are called the **variation morphisms** of the T-array.

From this definition we deduce the definition of two pairings between C and $\tilde{C}$, which make these two complexes dual. The pairings are the Seifert forms of $\mathcal{T}$.

Namely, for any $a \in H_p(C)$ and any $\tilde{b} \in H^p(\tilde{C})$,

$$\begin{aligned} S_1(a,\tilde{b}) &= \langle \mathrm{var}^{-1}(a), \tilde{b}\rangle, \\ S_2(\tilde{b},a) &= \langle \mathrm{var}^{*-1}(\tilde{b}), a\rangle, \end{aligned}$$

where var is the morphism induced by v in homologies.

Let $\mathcal{T}$ and $\mathcal{T}'$ be two T-arrays.

Definition 1.2: A **morphism** of arrays between $\mathcal{T}$ and $\mathcal{T}'$ is a pair $(\alpha,\tilde{\beta})$ of morphisms of complexes $\alpha\colon C \longrightarrow C'$ and $\tilde{\beta}\colon \tilde{C}' \longrightarrow \tilde{C}$ such that the following two diagrams are commutative:

$$\begin{array}{ccc} C' & \xleftarrow{v'} & D\tilde{C}' \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\tilde{\beta}^*} \\ C & \xleftarrow{v} & D\tilde{C} \end{array} \qquad\qquad \begin{array}{ccc} \tilde{C} & \xleftarrow{v^*} & DC \\ {\scriptstyle\tilde{\beta}}\uparrow & & \uparrow{\scriptstyle\alpha^*} \\ \tilde{C}' & \xleftarrow{v'^*} & DC' \end{array}$$

i.e. $\alpha v = v'\tilde{\beta}^*$ and $v^*\alpha^* = \tilde{\beta}v'^*$.

If there exists a morphism of arrays between two T-arrays $\mathcal{T}$ and $\mathcal{T}'$, we have the following proposition which gives the link between the Seifert forms of $\mathcal{T}$ and $\mathcal{T}'$.

PROPOSITION 1.3: *Let $(\alpha,\tilde{\beta})$ be a morphism of arrays between $\mathcal{T}$ and $\mathcal{T}'$. Then for any $a \in H_p(C)$ and any $\tilde{b}' \in H^p(\tilde{C}')$*

$$\begin{aligned} S_1(a,\tilde{\beta}(\tilde{b}')) &= S_1'(\alpha(a),\tilde{b}'), \\ S_2'(\tilde{b}',\alpha(a)) &= S_2(\tilde{\beta}(\tilde{b}'),a). \end{aligned}$$

Definition 1.4: Two $\top$-arrays $\mathcal{T}$ and $\mathcal{T}'$ are **isomorphic** if there exists a morphism of arrays $(\alpha, \tilde{\beta})\colon \mathcal{T} \longrightarrow \mathcal{T}'$ such that α and $\tilde{\beta}$ are quasi-isomorphisms.

If two arrays are isomorphic, then Proposition 1.3 implies that the Seifert forms of each array are the same through the isomorphism of arrays. So the Seifert forms are attached to the equivalence isomorphism class of arrays. This leads to the following definition of homological diagrams.

Definition 1.5: A **homological diagram** is an equivalence isomorphism class of $\top$-arrays; the Seifert forms associated to a homological diagram are defined by the Seifert forms of any of its representatives. It is denoted by

$$\mathcal{D} = \begin{array}{ccc} H_\bullet(C_\bullet) & \stackrel{\text{var}}{\longleftarrow} & H_\bullet(D\tilde{C}) \\ H^\bullet(\tilde{C}) & \stackrel{\text{var}^\bullet}{\longleftarrow} & H^\bullet(DC) \end{array}.$$

Definition 1.6: A **$\perp$-array** is a diagram $\mathcal{T}$ of the following type:

$$\begin{array}{ccc} D\tilde{C}^\bullet & \stackrel{w}{\longrightarrow} & C^\bullet \\ DC_\bullet & \stackrel{w^\bullet}{\longrightarrow} & \tilde{C}_\bullet \end{array}$$

where C is a complex of $\mathbb{Z}$-modules with differential of degree $+1$, and $\tilde{C}$ is a complex of $\mathbb{Z}$-modules with differential of degree -1, and w and $w^\bullet$ are quasi-isomorphisms.

The complexes C, $\tilde{C}$, DC, $D\tilde{C}$ are called the **vertices** of the $\perp$-array. The morphisms w and $w^\bullet$ are called the **variation morphisms** of the $\perp$-array.

From this definition we deduce the definition of two pairings (the Seifert forms) between $D\tilde{C}$ and DC, which make these two complexes dual.

Namely, for any $\tilde{f} \in H^p(D\tilde{C})$ and any $g \in H_p(DC)$,

$$\begin{aligned} \Sigma_1(\tilde{f}, g) &= \langle \text{war}(\tilde{f}), g \rangle, \\ \Sigma_2(g, \tilde{f}) &= \langle \text{war}^\bullet(g), \tilde{f} \rangle, \end{aligned}$$

where war is the morphism induced by w in homologies.

For $\perp$-arrays, as well as for $\top$-arrays, morphisms, isomorphisms and cohomological diagrams (as isomorphism classes of $\perp$-arrays) are defined.

1.2 Duality of arrays.

Definition 1.7: The **dual** of a $\top$-array $\mathcal{T}$,

$$\mathcal{T} = \begin{array}{ccc} C_\bullet & \stackrel{v}{\longleftarrow} & D\tilde{C}_\bullet \\ \tilde{C}^\bullet & \stackrel{v^\bullet}{\longleftarrow} & DC^\bullet \end{array}$$

is obtained by taking the duals of each of its vertices, with the dual corresponding variation morphisms. It is the following $\perp$-array $\mathcal{T}^*$:

$$\mathcal{T}^* = \begin{array}{ccc} DC^\bullet & \overset{w=v^*}{\longrightarrow} & \tilde{C}^\bullet \\ & & \\ D\tilde{C}_\bullet & \overset{w^*=v}{\longrightarrow} & C_\bullet \end{array}$$

Its isomorphism class is a **cohomological** diagram.

We can now give the definition of "dual" $\top$-arrays.

Definition 1.8: Two $\top$-arrays $\mathcal{T}$ and $\mathcal{T}'$ are **dual** if there exists a pair $(\alpha, \tilde{\beta})$ of quasi-isomorphisms of complexes α: $D\tilde{C} \longrightarrow C'$ and $\tilde{\beta}$: $\tilde{C}' \longrightarrow DC$ such that

$$\alpha^* = v^* \tilde{\beta} v'^*.$$

In this case, the two homological diagrams associated to $\mathcal{T}$ and $\mathcal{T}'$ are **dual**.

If two $\top$-arrays are dual, we get the following equalities for the Seifert forms (here $\Sigma_i'^*$ denotes the Seifert form of DC'):

$$\forall (a, \tilde{b}) \in H_p(C) \times H^p(\tilde{C}), \quad \Sigma_1'^*(\alpha^{*-1}(\tilde{b}), \tilde{\beta}^*(a)) = S_2(\tilde{b}, a).$$

A similar statement is true for $\Sigma_2'^*$ and S_1.

These equalities result from the following commutative diagram, where the first and last lines are the $\top$-array $\mathcal{T}$, and the two lines in the middle being the $\perp$-array $\mathcal{T}'^*$:

$$\begin{array}{ccc} \tilde{C} & \overset{v^*}{\longleftarrow} & DC \\ \alpha^* \uparrow & & \uparrow \tilde{\beta} \\ DC' & \overset{v'^*}{\longrightarrow} & \tilde{C}' \\ & (\mathcal{T}'^*) & \\ D\tilde{C}' & \overset{v'}{\longrightarrow} & C' \\ \tilde{\beta}^* \uparrow & & \uparrow \alpha \\ C & \overset{v}{\longleftarrow} & D\tilde{C}. \end{array}$$

2. Homological diagrams of singularities

A k-**corner isolated singularity** in $\mathbb{C}^{n+1}$ is defined by a k-**corner germ**, i.e. by a collection $(f, H_1, H_2, \ldots, H_{k+1})$, where a holomorphic germ f: $(\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0)$ defines an isolated singularity in $(\mathbb{C}^{n+1}, 0)$, and each H_i, for $1 \leq i \leq k+1$, is a hyperplane in $\mathbb{C}^{n+1}$ such that for any $1 \leq i_1 < \cdots < i_l \leq k+1$, the restriction $f_{|H_{i_1} \cap \cdots \cap H_{i_l}} = f_{i_1 \cdots i_l}$ is a germ of an isolated singularity in $(\mathbb{C}^{n-l+1}, 0)$.

In this section we construct homological diagrams for singularities. We begin with isolated singularities (sect. 2.1). Then we describe the case $k = 0$ (i.e. a boundary singularity) in detail (sect. 2.2), and the homological diagram for a corner singularity we get as a generalization of the preceding (sect. 2.3). For this construction see [3] and [12].

2.1 ISOLATED SINGULARITY. In [12] for any isolated hypersurface singularity given by a holomorphic germ $f\colon (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0)$, an array whose isomorphism class is the homological diagram associated to the singularity is constructed. We describe it now.

Let Θ be a triangulation of the Milnor fiber F of the germ f, compatible with the boundary ∂F, and Δ a barycentric subdivision of Θ. Let D be the cellular decomposition dual for Θ: if Θ_p is a p-simplex and a_p its barycenter, then the dual cell for a_p is the union of the $(2n-p)$-simplicies of Δ which meet Θ_p only in a_p.

Let h be the geometric monodromy of f.

The array $T_\Theta(f)$ associated to f and Θ is given by:

- $\mathcal{C}_{\Theta,\bullet}(f) = \mathcal{C}_{h(\Delta),\bullet}(F)$;
- $\tilde{\mathcal{C}}^{\bullet}_{\Theta}(f) = \mathcal{C}_{\Theta,2n-\bullet}(F)$, with the differential $\tilde{\partial}$ induced by ∂, the differential of $\mathcal{C}_{\Theta,\bullet}(F)$;
- the morphism v is the variation morphism which can be described in the context of simplicial complexes (see [12]).

If $g\colon (\mathbb{C}^{q+1}, 0) \longrightarrow (\mathbb{C}, 0)$ is another representative of the singularity given by f, then the array associated to g is isomorphic to the array associated to f, with a shift of $n - q$.

The homological diagram associated to the singularity given by f is the isomorphism class of $T_\Theta(f)$; it is described in degree n (the only interesting degree for this representative) by:

$$\mathcal{D}(f): \begin{array}{ccc} H_n(F) & \overset{\text{var}}{\longleftarrow} & H^n_c(F) \\ H_n(F) & \overset{\text{var}^*}{\longleftarrow} & H^n_c(F) \end{array}$$

where $\text{var} = \text{var}_f \circ \alpha$ is defined via the Alexander isomorphism α and the variation morphism var_f of the isolated hypersurface singularity given by the germ f:

$$\alpha\colon H^n_c(F) \longrightarrow H_n(F, \partial F); \quad \text{var}_f\colon H_n(F, \partial F) \longrightarrow H_n(F).$$

PROPOSITION 2.1: *Diagram $\mathcal{D}(f)$ is self-dual.*

Proof: First of all, we note that $(D\tilde{\mathcal{C}}_\Theta(f))_q = \mathcal{C}^{2n-q}_{\Theta,c}(F^\circ)$, where $F^\circ = F \setminus \partial F$.

Denote by T the tubular neighborhood of F° which is the union of the Δ simplicies which meet F°. Then the Alexander isomorphism maps $\mathcal{C}^{2n-q}_{\Theta,c}(F^\circ)$ onto $\mathcal{C}_{D,q}(T,\partial T)$. The Thom–Gysin morphism Γ (which is a quasi-isomorphism in this case) maps $\mathcal{C}_{D,q}(T,\partial T)$ onto $\mathcal{C}_{\Delta,q}(F^\circ)$, which is isomorphic to $\mathcal{C}_{h(\Delta),q}(F^\circ)$.

By a composition of these isomorphisms, we get the isomorphism between $D\tilde{\mathcal{C}}(f)$ and $\mathcal{C}(f)$.

By a similar (inverse) way, we get the isomorphism between $\tilde{\mathcal{C}}(f)$ and $D\mathcal{C}(f)$.

From the construction of variation morphism (see [12]), it is clear that the required commutation relationship is true. ∎

We suppose, for a given k-corner germ $(f, H_1, \ldots, H_{k+1})$, that a Milnor fibration compatible with the k-corner is given (see [12]). If F denotes the Milnor fiber of f, then $F_{i_1\cdots i_l} = F \cap H_{i_1} \cap \cdots \cap H_{i_l}$ is the Milnor fiber of $f_{i_1\cdots i_l}$.

In that context, let Θ be a triangulation of F compatible with ∂F and any of the $F_{i_1\cdots i_l}$, Δ a barycentric subdivision for Θ and D the cellular decomposition dual for Θ.

2.2 Boundary singularity case. In the case $k = 0$, we have the following construction which is based on [3] (see [12]).

The T-array $T_\theta(f, H_1)$ is defined by:

- $\mathcal{C}_{\Theta,\bullet}(f, H_1) = \mathcal{C}_{\Theta,\bullet}(f) \oplus \mathcal{C}_{\Theta,\bullet-1}(f_1)$;
- the differential for $\mathcal{C}_{\Theta,\bullet}(f, H_1)$ is given by:

$$\partial(a_0, a_1) = (\partial a_0 + (-1)^\bullet a_1, \partial a_1);$$

- $\tilde{\mathcal{C}}^\bullet_\Theta(f, H_1) = \tilde{\mathcal{C}}^\bullet_\Theta \oplus \tilde{\mathcal{C}}^{\bullet-1}_\Theta$;
- the differential $\tilde{\partial}$ for $\tilde{\mathcal{C}}^\bullet_\Theta(f, H_1)$ is given by:

$$\tilde{\partial}(\tilde{a}_0, \tilde{a}_1) = (\partial a_0, \partial a_1 + (-1)^\bullet \Gamma(\tilde{a}_0)),$$

  where Γ is the Thom–Gysin morphism (see [2] and [12]);
- the variation morphisms of $T_\theta(f, H_1)$ are induced by the variation morphisms of $T_\Theta(f)$ and $T_\Theta(f_1)$.

The homological diagram $\mathcal{D}(f, H_1)$ associated to the boundary singularity (f, H_1) is the isomorphism class of $T_\Theta(f, H_1)$.

We note that:

1. $\mathcal{C}(f, H_1)$ (resp. $\tilde{\mathcal{C}}(f, H_1)$) is the mapping cone of i: $\mathcal{C}(f_{|_{H_1}}) \longrightarrow \mathcal{C}(f)$ (resp. Γ: $\tilde{\mathcal{C}}(f) \longrightarrow \tilde{\mathcal{C}}(f_{|_{H_1}})$).
2. From the definition of $\mathcal{D}(f, H_1)$, it is possible to show (see [12]) that:
   - $H^n(\tilde{\mathcal{C}}_\Theta(f, H_1)) = H_n(F \setminus F \cap H_1)$,

- $H_n(\mathcal{C}_\Theta(f, H_1)) = H_n(F, F \cap H_1)$,

the other homology groups involved in the homological diagram being trivial.
3. There exist two natural morphisms

$$\delta: \mathcal{C}_{\Theta,\bullet-1}(f_1) \longrightarrow \mathcal{C}_{\Theta,\bullet}(f, H_1), \quad \tilde{\delta}: \tilde{\mathcal{C}}_\Theta^{\bullet-1}(f_1) \longrightarrow \tilde{\mathcal{C}}_\Theta^\bullet(f, H_1),$$

which define a morphism of arrays $T_\Theta(f, H_1) \to T_\Theta(f_1)[-1]$ (see [12]). Here $T[-1]$ denotes the array obtained from T by shifting by -1 the vertices and the variation morphisms of T.
4. There exist two natural morphisms

$$\gamma: \mathcal{C}_{\Theta,\bullet}(f) \longrightarrow \mathcal{C}_{\Theta,\bullet}(f, H_1), \quad \tilde{\gamma}: \tilde{\mathcal{C}}_\Theta^\bullet(f, H_1) \longrightarrow \tilde{\mathcal{C}}_\Theta^\bullet(f),$$

which define a morphism of arrays $T_\Theta(f) \to T_\Theta(f, H_1)$.

These morphisms of arrays induce the two exact sequences:

$$0 \to H_n(F) \to H_n(F, F \cap H_1) \to H_{n-1}(F \cap H_1) \to 0,$$
$$0 \leftarrow H_n(F) \leftarrow H_n(F \setminus F \cap H_1) \leftarrow H_{n-1}(F \cap H_1) \leftarrow 0.$$

In such a situation, we say that $\mathcal{D}(f, H_1)$ is an **extension** of $\mathcal{D}(f)$ and $\mathcal{D}(f_1)$.

More precisely, we get the following definition of extension of homological diagrams.

Definition 2.2: The homological diagram $\mathcal{D}$ is an **extension** of the two homological diagrams $\mathcal{D}'$ and $\mathcal{D}''$, we write $\mathcal{D} = \mathcal{E}(\mathcal{D}', \mathcal{D}'')$, if there are representatives $\mathcal{T}'$, $\mathcal{T}$ and $\mathcal{T}''$ of respectively $\mathcal{D}'$, $\mathcal{D}$ and $\mathcal{D}''$ such that there exist:

- a morphism of arrays $(\alpha', \tilde{\beta}'): \mathcal{T}' \longrightarrow \mathcal{T}$,
- a morphism of arrays $(\alpha'', \tilde{\beta}''): \mathcal{T} \longrightarrow \mathcal{T}''$,

such that $\mathrm{Im}(\alpha') \subset \ker(\alpha'')$, and $\mathrm{Im}(\tilde{\beta}'') \subset \ker(\tilde{\beta}')$.

From this definition, we get the following proposition:

PROPOSITION 2.3: $\mathcal{D}(f, H_1) = \mathcal{E}(\mathcal{D}(f), \mathcal{D}(f_1))$.

2.3 CORNER SINGULARITY CASE. The array $T_\theta(f, H_1, \ldots, H_{k+1})$ is defined by:

- $\mathcal{C}_{\Theta,\bullet}(f, H_1, \cdots, H_{k+1}) = \mathcal{C}_{\Theta,\bullet}(f) \bigoplus_{i=1}^{k+1} \mathcal{C}_{\Theta,\bullet-1}(f_i) \bigoplus_{1\le i<j\le k+1} \mathcal{C}_{\Theta,\bullet-2}(f_{i,j}) \bigoplus \cdots \bigoplus \mathcal{C}_{\Theta,\bullet-(k+1)}(f_{1,2,\ldots,k+1})$;
- $\tilde{\mathcal{C}}_\Theta^\bullet(f, H_1, \cdots, H_{k+1}) = \tilde{\mathcal{C}}_\Theta^\bullet(f) \bigoplus_{i=1}^{k+1} [\tilde{\mathcal{C}}_\Theta^{\bullet-1}] \bigoplus_{1\le i<j\le k+1} [\tilde{\mathcal{C}}_\Theta^{\bullet-2}(f_{i,j})] \bigoplus \cdots \bigoplus [\tilde{\mathcal{C}}_\Theta^{\bullet-(k+1)}(f_{1,2,\ldots,k+1})]$;
- the differentials for these two complexes are generalizations of the ones defined for a boundary singularity;
- the variation morphisms are induced by the variation morphisms of $T_\Theta(f)$, $T_\Theta(f_1), \ldots, T_\Theta(f_{1,2,\ldots,k+1})$.

The diagram $\mathcal{D}(f, H_1, , \dots, H_{k+1})$ associated to $(f, H_1, , \dots, H_{k+1})$ is the isomorphism class of $T_\Theta(f, H_1, \dots, H_{k+1})$.

Denote $H_{i_1 \cdots i_l} = H_{i_1} \cap \cdots \cap H_{i_l}$. We get the following proposition which is similar to Proposition 2.3.

PROPOSITION 2.4:

$$\mathcal{D}(f, H_1, \dots, H_{k+1}) = \mathcal{E}(\mathcal{D}(f, H_1, \dots, H_k), \mathcal{D}(f_{k+1}, H_{1,k+1}, \dots, H_{k,k+1})).$$

The same is true for any permutation on hyperplanes H_1 ,..., H_{k+1}.

3. Lagrange transformations and duality

3.1 LAGRANGE REFLECTIONS. Here we prove that there is a natural action of the group $\mathbb{Z}_2^{k+1}$ on the set of the k-corner isolated singularities. The action is generated by Lagrange reflections corresponding to the hyperplanes $H_1, \dots, H_{k+1}$ of the k-corner. In fact, the Lagrange reflections are involutions on the set of the k-corner isolated singularities rearranging the singularities of the decomposition. These results were announced in [10].

Let $(x_1, \dots, x_{k+1}, y_1, \dots, y_m) = (x, y)$, $k + m = n$, be coordinates in $\mathbb{C}^{n+1}$ such that the hyperplanes of the k-corner are the coordinate hyperplanes $H_i = \{x_i = 0\}, i = 1, \dots, k+1$. In this coordinate system, any k-corner germ f has the form $f(x_1, \dots, x_{k+1}, y)$. The decomposition of the k-corner singularity consists of 2^{k+1} ordinary singularities. Recall that for a k-corner isolated singularity, all the singularities of the decomposition are isolated.

Notation: To distinguish the k-corner singularity given by f from the ordinary one, we write $(f|k)$ for the k-corner singularity.

Definition 3.1: The **Lagrange reflection** $\mathcal{L}_j$ in the hyperplane H_j maps the k-corner germ $(f|k)$ to the new k-corner germ $\mathcal{L}_j(f|k) = (f^{*j}|k)$, where

$$f^{*j}(x_1, \dots, x_j^*, \dots, x_{k+1}, y, x_j) = f(x_1, \dots, x_{k+1}, y) + x_j x_j^*$$

is defined on $\mathbb{C}^{n+2}$ with coordinates $x_1, \dots, x_j^*, \dots, x_{k+1}, y, x_j$, and the k-corner in $\mathbb{C}^{n+2}$ is given by the hyperplanes

$$H_1, \dots, H_{j-1}, H_j^* = \{x_j^* = 0\}, H_{j+1}, \dots, H_{k+1}.$$

PROPOSITION 3.2 ([10]):

(i) *Any Lagrange reflection $\mathcal{L}_j, 1 \le j \le k+1$, gives a map on the set of the k-corner isolated singularities. In other words, if $(f|k) = (g|k)$, then $(f^{*j}|k) = (g^{*j}|k)$ for any $j = 1, \dots, k+1$.*

(ii) *For any $1 \leq j \leq k+1$, Lagrange reflection $\mathcal{L}_j$ rearranges the 2^{k+1} singularities of the decomposition of k-corner isolated singularity $(f|k)$:*

$$f^{*j}_{i_1 \cdots j \cdots i_l} \sim_{st} f_{i_1 \cdots \hat{j} \cdots i_l}, \quad f^{*j}_{i_1 \cdots \hat{j} \cdots i_l} \sim_{st} f_{i_1 \cdots j \cdots i_l},$$

where $\sim_{st}$ means stable equivalence of function germs. In other words, $\mathcal{L}_j$ either adds index j to the subscript (if there is no j) or removes this index from the subscript (if there is index j).

(iii) *Any Lagrange reflection $\mathcal{L}_j$, $1 \leq j \leq k$, is an involution on the set of the k-corner isolated singularities. In other words, $(f|k) = (\mathcal{L}^*_j(f^{*j})|k)$, where $\mathcal{L}^*_j$ is the Lagrange reflection in the hyperplane H^*_j.*

The statement (i) follows from Definition 3.1, the proof of (iii) is similar to the proof of Theorem 4.3 (see below), and statement (ii) is based on the following lemma.

LEMMA 3.3: *Let $g(x, y_1, \dots, y_m) = g(x, y)$ be a holomorphic germ at an isolated critical point $O \in \mathbb{C}^{m+1}$ with critical value 0. If $g(0, y_1, \dots, y_m) = g(0, y)$ also has an isolated critical point at the origin, then the germs $g(0, y)$ and $g(x, y) + xz$ are stable equivalent.*

Proof: We can write g in the form $g(x, y) = g(0, y) + xh(x, y)$ for some holomorphic germ h. Then we get

$$g(x, y_1, \dots, y_m) + xz = g(0, y) + x(z + h) = g(0, y) + xZ \sim_{st} g(0, y),$$

where $Z = z + h(x, y_1, \dots, y_m)$. ∎

Consider now the group generated by the reflections $\mathcal{L}_1, \dots, \mathcal{L}_{k+1}$. Any non-unit element of this group has the form

$$\mathcal{L}_{i_1 \dots i_l} = \mathcal{L}_{i_1} \cdot \ldots \cdot \mathcal{L}_{i_l}, \quad 1 \leq i_1 < \cdots < i_l \leq k+1.$$

This gives the transformation

$$\mathcal{L}_{i_1 \cdots i_l}(f) = \mathcal{L}_{i_1}(\mathcal{L}_{i_2}(\cdots(\mathcal{L}_{i_l}(f)\cdots) = f + x_{i_1} x^*_{i_1} + \cdots + x_{i_l} x^*_{i_l},$$

which is an involution on the set of the k-corner isolated germs. The decomposition of $(\mathcal{L}_{i_1 \cdots i_l}(f)|k)$ is obtained from the decomposition of $(f|k)$ by a permutation. Thus we get the following result ([10]).

THEOREM 3.4: *The Lagrange reflections $\{\mathcal{L}_j, 1 \leq j \leq k+1\}$ generate an action of the group $\mathbb{Z}_2^{k+1}$ on the set of the k-corner isolated singularities.*

3.2 THE ACTION OF THE LAGRANGE TRANSFORMATION ON HOMOLOGICAL DIAGRAMS. In this section, we study the product of the Lagrange reflections in all the hyperplanes of the k-corner, which is the **Lagrange transformation** $L = \mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_{k+1}$. We prove that the homological diagrams corresponding to k-corner singularities $(f|k)$ and $L(f|k)$ are dual.

For the Lagrange transformation, the results of sect. 3.1 give the following proposition.

PROPOSITION 3.5: *Let $(f, H_1, \ldots, H_{k+1})$ and $(f^*, H_1^*, \ldots, H_{k+1}^*)$ be a k-corner singularity and its Lagrange transform through L. Then*

$$f_{i_1 \cdots i_l} \sim_{st} f^*_{1 \cdots \hat{i}_1 \cdots \hat{i}_l \cdots k+1}, \quad 1 \leq i_1 < \cdots < i_l \leq k+1.$$

Now we can prove that the homological diagrams of a k-corner singularity and its Lagrange transform are dual, up to a shift.

THEOREM 3.6:

$$\mathcal{D}(f, H_1, \ldots, H_{k+1}) \text{ is dual to } \mathcal{D}(f^*, H_1^*, \ldots, H_{k+1}^*)[-1-k].$$

Proof: Proposition 3.5 implies that there exists an isomorphism of the arrays $T(f_{i_1 \cdots i_l})$ and $T(f^*_{1 \cdots \hat{i}_1 \cdots \hat{i}_l \cdots k+1})[-l]$ for any $1 \leq i_1 < \cdots < i_l \leq k+1$. The isomorphism is induced by the product of successive suspensions, and the triangulations of the successive Lagrange transforms can be taken as the image by these suspensions of the compatible triangulation chosen for $(f, H_1, \ldots, H_{k+1})$.

So, $\mathcal{C}(f_{i_1 \cdots i_l})$ is isomorphic to $\mathcal{C}(f^*_{1 \cdots \hat{i}_1 \cdots \hat{i}_l \cdots k+1})[-l]$, and by Proposition 2.1 it is isomorphic to $D\tilde{\mathcal{C}}(f^*_{1 \cdots \hat{i}_1 \cdots \hat{i}_l \cdots k+1})[-l]$. Also $\tilde{\mathcal{C}}(f_{i_1 \cdots i_l}) \cong \tilde{\mathcal{C}}(f^*_{1 \cdots \hat{i}_1 \cdots \hat{i}_l \cdots k+1})[-l]$, and by Proposition 2.1 is isomorphic to $D\mathcal{C}(f^*_{1 \cdots \hat{i}_1 \cdots \hat{i}_l \cdots k+1})[-l]$.

Thus we get the required isomorphisms between the vertices of the arrays $T(f, H_1, \ldots, H_{k+1})$ and $DT(f^*, H_1^*, \ldots, H_{k+1}^*)[-l]$. Indeed, the dual of any mapping cone of any $\varphi\colon \mathcal{C} \longrightarrow \mathcal{C}'$ is the mapping cone of $\varphi^*\colon D\mathcal{C}' \longrightarrow D\mathcal{C}$.

By construction, the commutation relationship for variation morphisms is true.

∎

4. Flag singularities

This section is devoted to another generalization of boundary singularities, namely to the singularities on a complex space with a flag.

We define flag singularities and describe the Lagrange transformation in this case. Then we discuss the homological diagrams of flag singularities and show that the Lagrange transformation of flag singularities gives a geometric realization of the duality for the homological diagrams associated to the flag singularities.

4.1 LAGRANGE TRANSFORMATION OF FLAG SINGULARITIES. Consider a complete flag of complex spaces in $\mathbb{C}^k$:

$$O \in \mathbb{C}^1 \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^{k-1}.$$

The direct product with $\mathbb{C}^m$ gives a **k-flag** $\mathcal{F}_k$ in $\mathbb{C}^{k+m}$:

$$\mathbb{C}^m \subset \mathbb{C}^1 \times \mathbb{C}^m \subset \cdots \subset \mathbb{C}^{k-1} \times \mathbb{C}^m.$$

Let now $(\mathbb{C}^n, \mathcal{F}_k)$ be a complex space $\mathbb{C}^n$ with some fixed k-flag

$$\mathcal{F}_k = \{L^0 \subset L^1 \subset \cdots \subset L^{k-1}\} \subset \mathbb{C}^n,$$

where $\dim L^i = i + m$, $0 \leq i \leq k-1$, $k + m = n$. A holomorphic germ $f: (\mathbb{C}^n, \mathcal{F}_k, O) \longrightarrow (\mathbb{C}, 0)$ we call a **k-flag germ**. Its stabilization is a k-flag germ $\tilde{f}(x,u) = f(x) + Q(u)$ on a space $\mathbb{C}^n \times \mathbb{C}^l$ with k-flag

$$\tilde{\mathcal{F}}_k = \{\tilde{L}^i = L^i \times \mathbb{C}^l, \ \ 0 \leq i \leq k-1\}.$$

Here $x = (x_1, \ldots, x_n)$ are coordinates in $\mathbb{C}^n$, $u = (u_1, \ldots, u_l)$ are coordinates in $\mathbb{C}^l$, Q is a Morse function.

Definition 4.1: A **k-flag singularity** $(f, \mathcal{F}_k)$ is a k-flag germ at a critical point,

$$f: (\mathbb{C}^n, \mathcal{F}_k, O) \longrightarrow (\mathbb{C}, 0),$$

considered up to diffeomorphisms of $\mathbb{C}^n$ preserving the flag $\mathcal{F}_k$ and up to stabilizations. A k-flag singularity $(f, \mathcal{F}_k)$ is **isolated** if all germs $f, f_i = f|_{L^i}$, $0 \leq i \leq k-1$ have isolated critical points at O. The set of (ordinary) singularities $(f_0, \ldots, f_{k-1}, f)$ is called the **decomposition** of the k-flag singularity $(f, \mathcal{F}_k)$.

Now we define the **Lagrange transformation** on the set of k-flag germs.

For a given k-flag germ $(f, \mathcal{F}_k)$, we introduce in $\mathbb{C}^n$ coordinates $x_1, \ldots, x_k$, $u_1, \ldots, u_m$, $k + m = n$, such that the k-flag $\mathcal{F}_k$ is given by

$$\mathcal{F}_k = \{\{x_1 = 0\} \supset \{x_1 = x_2 = 0\} \supset \cdots \supset \{x_1 = \cdots = x_k = 0\}\}.$$

Then the **Lagrange transform** of $(f, \mathcal{F}_k)$ is k-flag germ $(f^*, \mathcal{F}_k^*)$, where

$$f^*(x^*, x, u) = x_1^* x_1 + \cdots + x_k^* x_k - f(x, u)$$

is a germ on $\mathbb{C}^{n+k}$ with coordinates $x_1^*, \ldots, x_k^*, x_1, \ldots, x_k, u_1, \ldots, u_m$ and k-flag

$$\mathcal{F}_k^* = \{\{x_k^* = 0\} \supset \{x_k^* = x_{k-1}^* = 0\} \supset \cdots \supset \{x_k^* = \cdots = x_1^* = 0\}\}.$$

For k-flag singularities, we get the following proposition which is similar to Proposition 3.2.

PROPOSITION 4.2: *The Lagrange transformation defines a map on the set of k-flag singularities. If $(f^*, \mathcal{F}_k^*)$ is the Lagrange transform of the k-flag singularity $(f, \mathcal{F}_k)$, then $f_i \sim_{st} f_{k-i}^*$, $0 \le i \le k$.*

The first part of the proposition is a direct consequence of the definitions; the second part results from Lemma 3.3.

THEOREM 4.3: *On the set of all isolated k-flag singularities, the Lagrange transformation is an involution exchanging the order of the singularities in the decomposition.*

Proof: We need to check that k-flag germs $(f, \mathcal{F}_k)$ and $(f^{**}, \mathcal{F}_k^{**})$ define the same k-flag singularity. We have

$$f^{**}(x^{**}, x^*, x, u) = x_1^{**}x_1^* + \cdots + x_k^{**}x_k^* - x_1^*x_1 - \cdots - x_k^*x_k + f(x, u).$$

Consider k-flag germs $f^{**}(x^{**}, x^*, x, u)$ and $f(x^{**}, u)$ on the spaces with coordinates x^{**}, x^*, x, u and x^{**}, u respectively and k-flags given by the same equations in these spaces:

$$\{x_1^{**} = 0\} \supset \cdots \supset \{x_1^{**} = \cdots = x_k^{**} = 0\}.$$

We have to prove that germs $f^{**}(x^{**}, x^*, x, u)$ and $f(x^{**}, u)$ are stable equivalent. We get

$$\begin{aligned} f^{**}(x^{**}, x^*, x, u) = x_1^*(x_1^{**} - x_1) + \cdots &+ x_k^*(x_k^{**} - x_k) \\ &+ (f(x, u) - f(x^{**}, u)) + f(x^{**}, u). \end{aligned}$$

According to a version of Bézout theorem,

$$f(x, u) - f(x^{**}, u) = (x_1 - x_1^{**})\phi_1 + \cdots + (x_k - x_k^{**})\phi_k,$$

where ϕ_i, $1 \le i \le k$ are some holomorphic germs on x^{**}, x, u. Thus

$$\begin{aligned} f^{**}(x^{**}, x^*, x, u) - f(x^{**}, u) &= (x_1^{**} - x_1)(x_1^* - \phi_1) + \cdots + (x_k^{**} - x_k)(x_k^* - \phi_k) \\ &= X_1X_1^* + \cdots + X_kX_k^*, \end{aligned}$$

where $X_i = x_i^{**} - x_i$, $X_i^* = x_i^* - \phi_i$, $i = 1, \ldots, k$. This is a Morse function in variables $X_1, \ldots, X_k$, $X_1^*, \ldots, X_k^*$. It is clear that the diffeomorphism

$$x^{**} \longrightarrow x^{**}, \quad u \longrightarrow u, \quad x \longrightarrow X, \quad x^* \longrightarrow X^*$$

preserves the flag. Thus k-flag germs $(f^{**}, \mathcal{F}^{**})$ and $(f, \mathcal{F})$ define the same flag singularities. ∎

4.2 Diagrams and duality for flag singularities. Here, to any k-flag singularity we associate a homological diagram. As in the case of the corner singularities, we work with a Milnor fibration compatible with the flag. This means that if F is the Milnor fiber of f, then $F_i = F \cap L^i$ is the Milnor fiber for the singularity f_i.

As in section 1, we choose a triangulation Θ of F compatible with ∂F and any of the F_i, a barycentric subdivision Δ for Θ and the cellular decomposition D dual for Θ.

The ⊤-array $T_\Theta(f, \mathcal{F}_k)$ is defined by:

- $\mathcal{C}_{\Theta,\bullet}(f, \mathcal{F}_k) = \mathcal{C}_{\Theta,\bullet}(f) \oplus \mathcal{C}_{\Theta,\bullet-1}(f_1) \oplus \cdots \oplus \mathcal{C}_{\Theta,\bullet-k}(f_k)$ with the differential
$$\partial(a_0, a_1, \ldots, a_k) = (\partial a_0 + (-1)^\bullet a_1, \partial a_1 + (-1)^{\bullet-1} a_2, \ldots, \partial a_k);$$
- $\tilde{\mathcal{C}}_\Theta^\bullet(f, \mathcal{F}_k) = \tilde{\mathcal{C}}_\Theta^\bullet(f) \oplus \tilde{\mathcal{C}}_\Theta^{\bullet-1}(f_1) \oplus \cdots \oplus \tilde{\mathcal{C}}_\Theta^{\bullet-k}(f_k)$ with the differential $\partial(\tilde{a}_0, \tilde{a}_1, \cdots, \tilde{a}_k) = (\partial a_0, \partial a_1 + (-1)^\bullet \Gamma(a_0), \ldots, \partial a_k + (-1)^{\bullet-k+1}\Gamma(a_{k-1}))$, where Γ is the Thom–Gysin morphism;
- the variation morphisms are induced by the variation morphisms of $T_\Theta(f)$, $T_\Theta(f_1), \ldots, T_\Theta(f_k)$.

Definition 4.4: The homological diagram $\mathcal{D}(f, \mathcal{F}_k)$ associated to the k-flag singularity $(f, \mathcal{F}_k)$ is the isomorphism class of $T_\Theta(f, \mathcal{F}_k)$.

The following proposition describes a k-flag singularity as an extension of an ordinary singularity and a $(k-1)$-flag singularity.

Proposition 4.5: *$\mathcal{D}(f, \mathcal{F}_k) = \mathcal{E}(\mathcal{D}(f), \mathcal{D}(f_{k-1}, \mathcal{F}_{k-1}))$, where $(k-1)$-flag $\mathcal{F}_{k-1}$ is obtained from $\mathcal{F}_k$ by deleting L^{k-1}.*

This proposition is a direct consequence of the definitions.

Theorem 4.6: *If k-flag singularity $(f^*, \mathcal{F}_k^*)$ is the Lagrange transform of $(f, \mathcal{F}_k)$, then the diagrams $\mathcal{D}(f, \mathcal{F}_k)$ and $\mathcal{D}(f^*, \mathcal{F}_k^*)[-k]$ are dual.*

Proof: This proof is similar to the proof of Theorem 3.6. Proposition 4.2 implies that there is an isomorphism between $T(f_l)$ and $T(f_{k-l}^*)[-l]$. This induces, via

Proposition 2.1, the required isomorphisms, as the dual of any mapping cone of any $\varphi\colon C \longrightarrow C'$ is the mapping cone of $\varphi^*\colon DC' \longrightarrow DC$. ∎

References

[1] V. I. Arnold, *Critical points of functions on a manifold with a boundary, the simple Lie groups B_k, C_k, F_4 ...*, Russian Mathematical Surveys **33** (1978), 91–105.

[2] J.-P. Brasselet, *Définition combinatoire des homomorphismes de Poincaré, Alexander et Thom pour une pseudo variété*, Asterisque **82–83** (1981), 71–91.

[3] J. A. Carlson, *Polyhedral resolutions of algebraic varieties*, Transactions of the American Mathematical Society **292** (1985), 595–612.

[4] N. T. Dai, N. H. Duc and F. Pham, *Singularités nondégénérées des systémes de Gauss-Manin*, Memoire de la Societé Mathématique de France, Novelle Serie **6** (1981), 85 pp.

[5] A. S. Kryukovskii and D. V. Rastiagaev, *Classification of unimodal and bimodal corner singularities*, Funktsional'nyi Analiz i ego Prilozheniya **26** (1992), 213–215.

[6] I. Scherback, *Duality of boundary singularities*, Russian Mathematical Surveys **39** (1984), 195–196.

[7] I. Scherback, *Singularities in presence of symmetries*, American Mathematical Society Translations, Series 2 (Topics in Singularity Theory) **180** (1997), 189–196.

[8] I. Scherbak, *Symmetric and flag singularities*, in preparation.

[9] I. Scherback and A. Szpirglas, *Boundary singularities: topology and duality*, Advances in Soviet Mathematics **21** (1994), 213–223.

[10] D. Siersma, *Singularities of functions on boundaries, corners, etc.*, Quarterly Journal of Mathematics. Oxford. Second Series **32** (1981), 119–127.

[11] A. Szpirglas, *Singularités de bord: dualité, formules de Picard Lefschetz relatives et diagrammes de Dynkin*, Bulletin de la Société Mathématique de France **118** (1990), 451–486.

[12] A. Szpirglas, *Diagrammes homologiques de variation et singularités de coin*, Prépublication mathématique, Université Paris Nord, **97-11** (1997).

[13] V. M. Zakalukin, *Singularities of circle contact with surface and flags*, Functional Analysis and its Applications **31** (1997), 73–76.